

# Effects of inhomogeneous activity of players and noise on cooperation in spatial public goods games

Jian-Yue Guan,<sup>1,\*</sup> Zhi-Xi Wu,<sup>1,2,†</sup> and Ying-Hai Wang<sup>1,‡</sup>

<sup>1</sup>*Institute of Theoretical Physics, Lanzhou University, Lanzhou Gansu 730000, China*

<sup>2</sup>*Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong, China*

(Received 6 July 2007; revised manuscript received 10 October 2007; published 2 November 2007)

We study the public goods game in the noisy case by considering the players with inhomogeneous activity of teaching on a square lattice. It is shown that the introduction of the inhomogeneous activity of teaching the players can remarkably promote cooperation. By investigating the effects of noise on cooperative behavior in detail, we find that the variation of cooperator density  $\rho_C$  with the noise parameter  $\kappa$  displays several different behaviors:  $\rho_C$  monotonically increases (decreases) with  $\kappa$ ;  $\rho_C$  first increases (decreases) with  $\kappa$  and then it decreases (increases) monotonically after reaching its maximum (minimum) value, which depends on the amount of the multiplication factor  $r$ , on whether the system is homogeneous or inhomogeneous, and on whether the adopted updating is synchronous or asynchronous. These results imply that the noise plays an important and nontrivial role in the evolution of cooperation.

DOI: [10.1103/PhysRevE.76.056101](https://doi.org/10.1103/PhysRevE.76.056101)

PACS number(s): 89.65.-s, 02.50.Le, 07.05.Tp, 87.23.Kg

## I. INTRODUCTION

Cooperation plays an important role in the real world, ranging from biological systems to economic and as well as social systems [1]. Scientists from different fields of natural and social sciences often resort to evolutionary game theory [2,3] as a common mathematical framework and the prisoner's dilemma game (PDG) as a metaphor for studying cooperation between selfish and unrelated individuals [3]. The PDG, which captures typical pairwise interactions in many cases, has become the leading paradigm to explain cooperative behavior [4,5]. As an alternative, the public goods game (PGG), which can be regarded as a PDG with more than two participants, attracts also much attention to study the emergence of cooperative behavior [6]. In a typical PGG played by  $N$  agents [7], each of them must decide independently and simultaneously whether or not to invest in a common pool. The collected investment will be multiplied by a factor  $r$ , and then is redistributed uniformly among all players irrespective of their actual contributions. It was shown that for the values of  $1 < r < N$ , the *free riders* or *defectors* (i.e., those who do not invest) will dominate the whole population [8,9].

To provide an escape hatch out of economic stalemate, Hauert *et al.* have introduced the voluntary participation in such public goods and found that it results in a substantial and persistent willingness to cooperate [10]. Szabó *et al.* have studied the voluntary participation in PGG on a square lattice [7] and found that the introduction of loners leads to a cyclic dominance of the strategies and promotes substantial levels of cooperation. A remarkable increase of cooperation is also observed for those systems where the inhomogeneous imitation activity is introduced artificially to characterize the asymmetric and different influence of players to each other [11]. In Ref. [12], Wu *et al.* have studied the PDG with the

dynamic preferential rule and found that cooperation is substantially enhanced due to this simple selection mechanism. The effect of payoffs and noise on the maintenance of cooperative behavior has been studied in PDG where two types of players ( $A$  and  $B$ ) with different activity of teaching are introduced on regular connectivity structures [13]. It was found that the introduction of the inhomogeneous activity of teaching the players can remarkably promote cooperation [13].

In the present paper, we study the effects of noise on the cooperative behavior by considering the players with the inhomogeneous activity of teaching in PGG on the square lattice. Our goals are to investigate (i) whether the inhomogeneous activity of teaching the players can promote cooperation in PGG, and (ii) how the noise level affects the cooperative behavior in the PGG. Our observations suggest that the cooperative behavior is remarkably enhanced due to the introduction of the inhomogeneous activity of teaching. And we also find that the cooperator density  $\rho_C$  changing with the noise level exhibits several different behaviors, which depends on the values of the multiplication factor  $r$ , on whether the system is homogeneous or inhomogeneous, and also on whether the adopted updating is synchronous or asynchronous.

## II. THE MODEL AND SIMULATIONS

Motivated by previous research work [12,13], we consider in our model two types of players ( $A$  and  $B$ ) who are distributed randomly on the sites of a square lattice before the start of the simulation. The fraction of players  $A$  and  $B$  are denoted by  $\nu$  and  $(1-\nu)$ , respectively. We consider the compulsory version of a spatial PGG [7], i.e., both types of players are only cooperators ( $C$ ) or defectors ( $D$ ). Each player interacts only with its four nearest neighbors (von Neumann neighborhood). Thus the size of interaction is  $g=5$ . The payoff of each player depends on the number of cooperators  $n_c$  in the neighborhood, i.e., the player  $x$ 's payoff is

\*guanjr03@lzu.cn

†wupiao2004@yahoo.com.cn

‡yhwang@lzu.edu.cn

$$\begin{aligned}
 p_x &= \frac{rn_c}{g} - 1, \quad \text{if } s_x = C, \\
 p_x &= \frac{rn_c}{g} - 0, \quad \text{if } s_x = D,
 \end{aligned}
 \tag{1}$$

where  $s_x$  denotes the strategy of the player  $x$ , and  $r$  represents the multiplication factor on the public good. The cooperative investments are normalized to unity and  $r > 1$  must hold such that groups of cooperators are better off than groups of defectors.

After each generation, the players try to maximize their individual payoff by imitating (learning) one of the more successful neighboring strategies. Following previous studies [5,7,12–14], player  $x$  will adopt the randomly chosen neighbor  $y$ 's strategy  $s_y$  with a probability depending on the payoff difference ( $p_x - p_y$ ) as

$$W(s_x \rightarrow s_y) = \omega_{xy} \frac{1}{1 + \exp[(p_x - p_y)/\kappa]},
 \tag{2}$$

where  $\kappa$  denotes the amplitude of noise. The prefactor  $\omega_{xy}$  is given as

$$\begin{aligned}
 \omega_{xy} &= 1, \quad \text{if } n_y = A, \\
 \omega_{xy} &= \omega, \quad \text{if } n_y = B,
 \end{aligned}
 \tag{3}$$

where the value of  $\omega$  ( $0 < \omega < 1$ ) characterizes the strength of reduced teaching activity if the site  $y$  is occupied by a player of type  $B$  [13].

Simulations were carried out for a population of  $N = 100 \times 100$  individuals. We study the key quantity of cooperator density  $\rho_C$  in the steady state. Initially, the two strategies of  $C$  and  $D$  are randomly distributed among the individuals with equal probability  $1/2$ . The above model was simulated with both synchronous and asynchronous updating. Eventually, the system reaches a dynamic equilibrium state. The simulation results were obtained by averaging over the last  $10^4$  Monte Carlo (MC) time steps of the total  $10^5$ . Each data point results from an average of over 20 realizations.

### III. SIMULATION RESULTS AND ANALYSIS

To facilitate comparison, we first consider the homogeneous system ( $\nu = 1$ ), i.e., the system for two types of players with the same teaching activity. Figure 1 shows that the cooperator density  $\rho_C$  varies as the multiplication factor  $r$  when  $\nu = 1$  by using synchronous and asynchronous updating. We can find from this figure that, below some threshold values of the multiplication factor ( $r < r_s$  for synchronous updating and  $r < r_{as}$  for asynchronous updating), the fraction of cooperators  $\rho_C$  vanishes and the defectors reign; while for the values of multiplication factor  $r > r_s$  and  $r > r_{as}$ , the cooperator density  $\rho_C$  increases with  $r$ . For large  $r$ ,  $\rho_C$  can reach its maximum value 1, i.e., the system reaches the absorbing state of all cooperators, because large values of  $r$  are in favor of cooperators in the context of the PGG [8]. We compare the simulation results for synchronous and asynchronous updat-

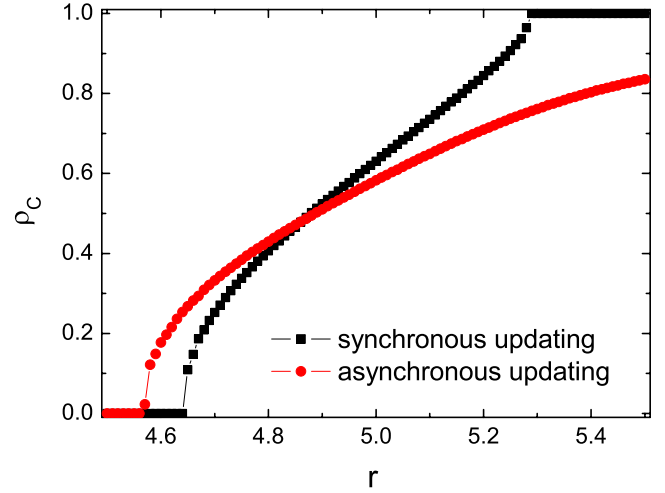


FIG. 1. (Color online) The cooperator density  $\rho_C$  vs the multiplication factor  $r$  when  $\nu = 1$  by using the synchronous and asynchronous updating. The noise parameter is  $\kappa = 0.1$ .

ing and find that, when  $r$  is smaller than a certain value  $r_c$ ,  $\rho_C$  is larger for asynchronous updating than that for synchronous updating, but the reverse is true for  $r > r_c$ . Here we want to remark that in the process of our simulation, for synchronous updating, in each time step, all sites are updated simultaneously through competition with a randomly chosen neighbor. For asynchronous updating, however, in each time step, the individuals update their strategies one by one in a random sequence. It is known that in the context of the PGG, small values of  $r$  favor defectors and large  $r$  benefits cooperators [8,9]. Comparing the average frequency of cooperators under the same values of  $r$ , we find that for small (large)  $r$ , defectors (cooperators) live better for synchronous updating than that for asynchronous updating.

In Fig. 2, we show the variation of cooperator density  $\rho_C$  with the fraction  $\nu$  of individuals  $A$  for several different values of  $\omega$  when  $r = 4.8$  and  $\kappa = 0.1$  by using synchronous (solid symbols) and asynchronous (hollow symbols) updating. In our model, if  $\omega = 1$  the system is equivalent to a homogeneous system (discussed in Fig. 1). For  $\omega < 1$ , i.e., the inho-

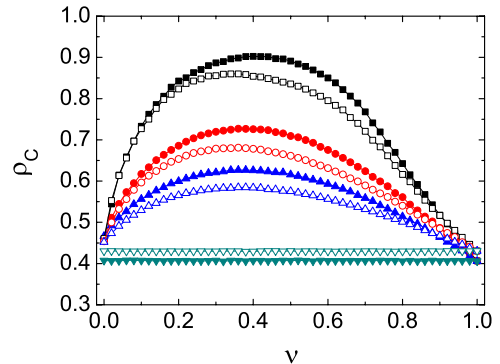


FIG. 2. (Color online) The simulation results of cooperator density  $\rho_C$  as a function of  $\nu$  for different values of  $\omega$  when  $r = 4.8$  and  $\kappa = 0.1$  by using synchronous (solid symbols) and asynchronous (hollow symbols) updating on the square lattice. The values of  $\omega$  from top to bottom are, respectively, 0.01, 0.1, 0.2, and 1.0.

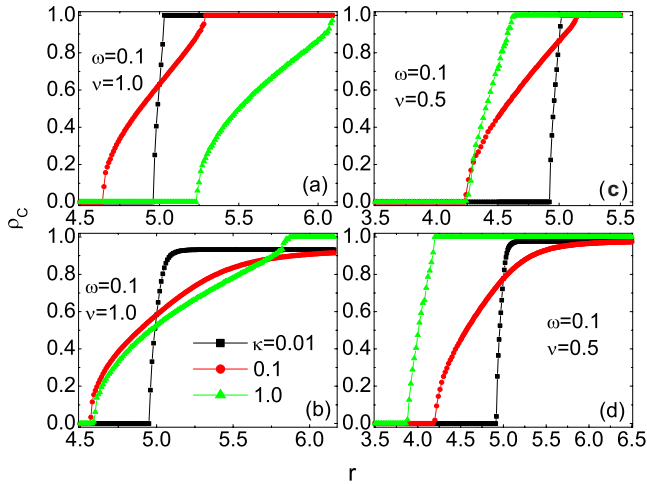


FIG. 3. (Color online) The cooperator density  $\rho_C$  vs the multiplication factor  $r$  in homogeneous (a),(b) and inhomogeneous (c),(d) systems by using the synchronous (a),(c) and asynchronous (b),(d) updating for three noise parameters:  $\kappa=0.01, 0.1, 1.0$ .

homogeneous system, the cooperator density  $\rho_C$  increases monotonously until reaching the maximum value at  $\nu \approx 0.4$  ( $\nu \approx 0.33$ ) for synchronous (asynchronous) updating, and then it decreases with  $\nu$ . And we also find that for the same value of  $\nu$ , the smaller the  $\omega$  is, the larger the  $\rho_C$  will be. This phenomenon is similar to the results obtained in Ref. [13]. We argue that it is a general phenomenon that the introduction of reducing activity of teaching can remarkably enhance the cooperative behavior in the context of whether PDG or PGG [15]. From Fig. 2, we can also see that the qualitative results of the cooperative behavior are unchanged for both synchronous and asynchronous updating. Quantitatively, for the same value of  $\nu$ , the cooperator density  $\rho_C$  is larger for synchronous updating than that for asynchronous updating when  $\omega < 1$ , but the reverse is true in the case of  $\omega = 1$ . It indicates that the enhanced level of cooperation due to the introduction of reducing activity of teaching depends on the detailed updating (synchronous or asynchronous).

In our simulations, we find that the level of noise, i.e., the magnitude of  $\kappa$ , has a nontrivial effect on the evolution of cooperation. We show in Fig. 3 the cooperator density  $\rho_C$  varying with the multiplication factor  $r$  in homogeneous [Figs. 3(a) and 3(b)] and inhomogeneous [Figs. 3(c) and 3(d)] systems for several different values of  $\kappa$  by using the synchronous [Figs. 3(a) and 3(c)] and asynchronous [Figs. 3(b) and 3(d)] updating. In the stationary state  $\rho_C$  monotonously increases if  $r$  is increased for the same noise parameter. For the three noise parameters  $\kappa=0.01, 0.1, 1.0$  in homogeneous systems [Figs. 3(a) and 3(b)], when  $r=4.8$ ,  $\rho_C$  first increases and then decreases with  $\kappa$ , while for  $r=5.1$ ,  $\rho_C$  monotonously decreases with  $\kappa$ ; for larger values of  $r=6.0$ , the cooperator density displays different behaviors with  $\kappa$ :  $\rho_C$  monotonously decreases with  $\kappa$  for synchronous updating in Fig. 3(a), and  $\rho_C$  first decreases and then increases with  $\kappa$  for asynchronous updating in Fig. 3(b). In inhomogeneous systems [Figs. 3(c) and 3(d)], when  $r=4.8$ ,  $\rho_C$  monotonously increases with  $\kappa$ , while for  $r=5.05$ ,  $\rho_C$  first decreases and then increases with  $\kappa$ ; for smaller values of  $r=4.26$ , the co-

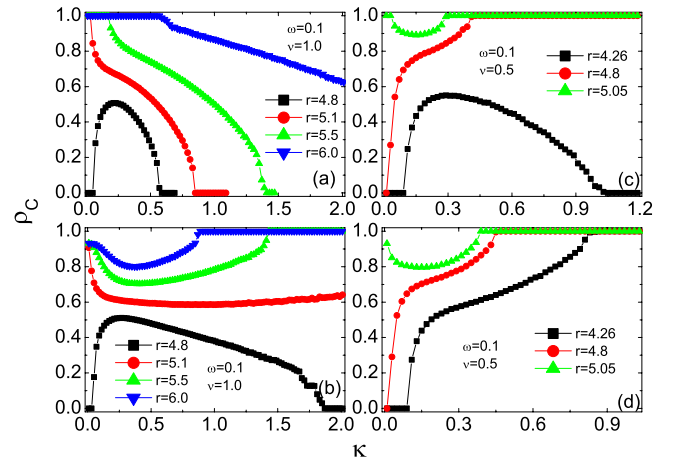


FIG. 4. (Color online) The cooperator density  $\rho_C$  vs the noise parameter  $\kappa$  in homogeneous (a),(b) and inhomogeneous (c),(d) systems by using the synchronous (a),(c) and asynchronous (b),(d) updating for different values of  $r$ : the detailed values are shown in the plots.

operator density displays different behaviors with  $\kappa$ :  $\rho_C$  first increases and then decreases with  $\kappa$  for synchronous updating in Fig. 3(c), and  $\rho_C$  monotonously increases with  $\kappa$  for asynchronous updating in Fig. 3(d). To investigate this in detail, we plot  $\rho_C$  vs  $\kappa$  for different values of  $r$  by using synchronous and asynchronous updating in homogeneous and inhomogeneous systems, respectively (see Fig. 4).

Before making further discussions about our MC data, we want to give a brief description of the mean field result for the PGG. In a well mixed population (i.e., in the mean field case) with initial number of cooperators  $N\rho_C$ , each cooperator and defector will get, respectively,  $r\rho_C - 1$  and  $r\rho_C$  payoff. If a defector (cooperator) changes its strategy to be a cooperator (defector), then its payoff variation is  $r/N - 1$  (and  $1 - r/N$ ). For any values of  $r$  larger than  $N$ , one can easily see that the transformation of defection to cooperation is preferred by the players, and the whole population will be dominated by cooperators. In the reverse, for  $1 < r < N$ , defectors dominate the population [5,8,9]. Looking back on Fig. 3, it is clear that in the low noise limit  $\kappa=0.01$ , the MC data are in accordance with the results predicted by mean field theory [5,8,9], i.e.,  $\rho_C=0$  if  $r < 5.0$  and  $\rho_C=1$  if  $r > 5.0$  ( $N=g=5$  in the present case).

Figure 4 shows that the fraction of cooperators  $\rho_C$  varies with the noise parameter  $\kappa$  for different values of  $r$  in the two systems [the homogeneous in Figs. 4(a) and 4(b), the inhomogeneous in Figs. 4(c) and 4(d)] by using two updating rules [the synchronous updating in Figs. 4(a) and 4(c), the asynchronous updating in Figs. 4(b) and 4(d)]. Let us first see the case in homogeneous systems. In Figs. 4(a) and 4(b), when  $r=4.8$ , the cooperator density  $\rho_C$  first increases monotonously until reaching the maximum value, and then decreases monotonously with  $\kappa$ , which indicates that the cooperative behavior can be remarkably enhanced at the optimal level of noise both for synchronous and asynchronous updating in homogeneous systems. For larger values of  $r$ , the concentration of cooperators  $\rho_C$  changing with  $\kappa$  displays different behaviors: It decreases monotonously with  $\kappa$  for

synchronous updating [see the curves for  $r=5.1, 5.5, 6.0$  in Fig. 4(a)]; while for asynchronous updating, when increasing the noise level, one can observe the valleys in the concentration of cooperators [see the curves for  $r=5.1, 5.5, 6.0$  in Fig. 4(b)]. That is, a large level of noise evidently inhibits the persistence of cooperation for synchronous updating; but for asynchronous updating, the cooperative behavior can be inhibited mostly at a certain value of  $\kappa$ , which indicates that the cooperative behavior depends remarkably on the updating rule. In inhomogeneous systems, for synchronous updating [Fig. 4(c)], the cooperator density  $\rho_C$  varying with  $\kappa$  displays three different behaviors:  $\rho_C$  first increases with  $\kappa$  and then decreases monotonously after reaching its maximum value for a small value of  $r=4.26$ ;  $\rho_C$  increases monotonously with  $\kappa$  for a moderate value of  $r=4.8$ ;  $\rho_C$  first decreases with  $\kappa$  and then increases monotonously after reaching its minimum value for a large value of  $r=5.05$ . This indicates that the changing behavior of  $\rho_C$  with  $\kappa$  is related closely to the multiplication factor  $r$ . But for asynchronous updating [Fig. 4(d)], when  $r=4.26$ ,  $\rho_C$  increases monotonously with  $\kappa$ , which is different from the case for synchronous updating. Once again, this indicates definitely that the cooperative behavior has strong dependence on the updating rule.

In addition, for the synchronous updating and the same value of  $r$ , we can observe a peak in the concentration of cooperators when increasing the noise level in homogeneous systems [see the curve for  $r=4.8$  in Fig. 4(a)], but the cooperator density  $\rho_C$  increases monotonously with the parameter  $\kappa$  in inhomogeneous systems [see the curve for  $r=4.8$  in Fig. 4(c)]. It indicates that the evolution of cooperation also depends on whether the system is homogeneous or inhomogeneous.

From the above behaviors, we can conclude that the effects of noise on the cooperative behavior remarkably depend on the values of  $r$ , on whether the system is homogeneous or inhomogeneous, and on whether the updating is synchronous or asynchronous.

#### IV. CONCLUSION

In summary, we have studied the effects of noise on cooperation in spatial public goods games by considering two types of players with different activity of teaching. It was shown that the introduction of the inhomogeneous activity of teaching the players can remarkably enhance the persistence of cooperation. The introduction of the inhomogeneous activity of teaching can partially resolve the dilemma of cooperation and may shed light on the evolution of cooperation in a society. By investigating the effects of noise on the cooperative behavior in detail, we found that the cooperator density  $\rho_C$  varying with the noise level  $\kappa$  displays several different behaviors, which remarkably depends on the values of  $r$ , on whether the system is homogeneous or inhomogeneous, and on whether the updating is synchronous or asynchronous. Interestingly, when increasing the level of noise, we observed both peaks and valleys in the concentration of cooperators for the middle level of noise. Thus the effect of noise is a correlated factor in game dynamics, which plays an important role in the evolution of cooperation.

#### ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 10775060.

- 
- [1] A. M. Colman, *Game Theory and Its Applications in the Social and Biological Sciences* (Butterworth-Heinemann, Oxford, 1995).
  - [2] J. M. Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, England, 1982).
  - [3] H. Gintis, *Game Theory Evolving* (Princeton University Press, Princeton, NJ, 2000).
  - [4] R. Axelrod and W. D. Hamilton, *Science* **211**, 1390 (1981).
  - [5] G. Szabó and G. Fáth, *Phys. Rep.* **446**, 97 (2007).
  - [6] J. H. Kagel and A. E. Roth, *The Handbook of Experimental Economics* (Princeton University Press, Princeton, NJ, 1997).
  - [7] G. Szabó and C. Hauert, *Phys. Rev. Lett.* **89**, 118101 (2002).
  - [8] C. Hauert and G. Szabó, *Complexity* **8**, 31 (2003).
  - [9] C. Hauert, M. Holmes, and M. Doebeli, *Proc. R. Soc. London, Ser. B.* **273**, 2565 (2006); C. Hauert, *Adv. Complex Syst.* **9**, 315 (2006).
  - [10] C. Hauert *et al.*, *Science* **296**, 1129 (2002).
  - [11] B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, *Phys. Rev. E* **66**, 021907 (2002).
  - [12] Z.-X. Wu *et al.*, *Chin. Phys. Lett.* **23**, 531 (2006); Z.-X. Wu, X. J. Xu, Z. G. Huang, S. J. Wang, and Y. H. Wang, *Phys. Rev. E* **74**, 021107 (2006).
  - [13] A. Szolnoki and G. Szabó, *Europhys. Lett.* **77**, 30004 (2007).
  - [14] G. Szabó and C. Tóke, *Phys. Rev. E* **58**, 69 (1998); G. Szabó, J. Vukov, and A. Szolnoki, *ibid.* **72**, 047107 (2005); C. Hauert and G. Szabó, *Am. J. Phys.* **73**, 405 (2005).
  - [15] Recently, we have found that the promotion of cooperation due to the inhomogeneous activity of the players also holds for other  $2 \times 2$  games, for example, the snowdrift game [5].